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ON THE APPLICATION OF THERMODYNAMICS TO THE THERMO-ELECTRIC CIRCUIT

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It has always been recognized that the application of the first and second laws of thermodynamics to the thermo-electric circuit by Kelvin, which gave the well-known relations:

$$P_{AB} = \tau \frac{dE_{AB}}{d\tau} \tag{1}$$

$$\sigma_B - \sigma_A = \tau \frac{d^2E_{AB}}{d\tau^2} \tag{2}$$

involved the assumption that the unavoidable irreversible processes of heat conduction and the generation of Joulean heat may be neglected in the argument. Kelvin himself expressly emphasized this point, and regarded the relations deduced by thermodynamic analysis as merely suggestive relations, which required the further check of experiment.[1] A great number of experimental attacks on the problem have been made—the experimental difficulties are not slight, and the experimental verification is not all that could be asked, but there are no known contradictions and with every improvement in experimental accuracy the verification becomes closer. As is to be expected, the verification is better for the formula giving the Peltier heat than for that giving the Thomson heat. The consensus of present opinion, as shown for example, by the article of Laski on thermo-electricity in the "Handbuch der Physik," vol. XIII, seems to be that, as a matter of fact, Kelvin's relations are very nearly correct, but that further experimental verification is desirable.

It is the purpose of this note to show that by a rearrangement of the argument the equations of Kelvin can be rigorously deduced from thermodynamic principles, without any assumption as to the justifiability of neglecting the effect of necessary irreversible processes.

The cycle to which Kelvin applied his argument was a direct one; a

couple with junctions at different temperatures was used as the source of e. m. f. by which an electromagnetic engine was operated, the couple thus supplying useful work by functioning as a thermodynamic engine between the temperatures of the two junctions as source and sink of heat. Even with the most favorable choice of the dimensions of the couple, it is not possible to make the reversible entropy changes at source and sink more than a very small fraction of the entropy changes unavoidably associated with the irreversible processes of heat conduction and Joulean heating, as was first shown by Boltzmann.

Instead of the direct cycle of Kelvin, consider the inverse cycle, in which a current is driven by an external e. m. f. through a couple of dissimilar metals, as is shown in figure 1. The junctions are supposed located within a heat insulating enclosure. The Peltier effect at the junctions will result in a generation of heat at one junction and an absorption at the other, so that one junction will rise in temperature and the other fall. Allow this warming and cooling to progress until the temperature difference no longer increases, but there is a steady flow of heat by conduction from one junction to the other exactly accounting for the Peltier heat at the junctions, to terms of the first order. The heat lost by conduction along A to the outside may be made vanishingly small by making A long in comparison with B. In addition to the potential difference arising from the iR drop, an additional external potential difference is required to drive the current against the counter e. m. f. arising from the difference of temperature between the two junctions. The difference between our set-up and that of Kelvin is that we deliberately so arrange matters that *all* the energy input is frittered away in the irreversible process of heat conduction from one temperature to another. This essentially irreversible process means an increase in the total entropy of the universe. If the final difference of temperature between the junctions is $\Delta\tau$, P_{AB} the Peltier heat, and τ the mean temperature of the junctions, then the increase of entropy per unit time per unit current is, by familiar elementary considerations, $\frac{P_{AB}\Delta\tau}{\tau^2}$, correct to first-order terms. Any difference of Peltier heat at the two junctions due to temperature difference, or generation of Thomson heat in the body of the metal, evidently contributes only second-order terms. The increase of entropy accompanying this essentially irreversible process must be located in the body of the thermocouple, and must be manifested as a gradual rise of temperature of the whole couple, for obviously there is no necessary increase in the

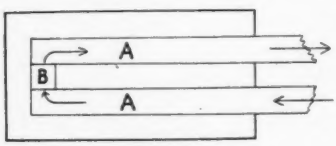

FIGURE 1

entropy of the external universe, since the electric current may be generated reversibly by a dynamo driven by a spring in which there are no thermal effects and therefore no entropy changes, and in the couple itself there are no possible chemical changes and only a rise of temperature to give an increase of entropy after the termination of the process. The increase of entropy of the couple can be found at once. The energy put into the couple by the electric current manifests itself eventually as a rise of temperature. The energy input per unit time per unit current is $\Delta\tau \frac{dE_{AB}}{d\tau}$, E_{AB} being the e. m. f. of the couple as in the formulas of Kelvin. The entropy increase due to this energy input is $\frac{\Delta\tau}{\tau}\frac{dE_{AB}}{d\tau}$. Equating this to the increase of entropy accompanying the irreversible heat conduction gives

$$\frac{P_{AB}\Delta\tau}{\tau^2} = \frac{\Delta\tau}{\tau}\frac{dE_{AB}}{d\tau},$$

whence at once

$$P_{AB} = \tau\frac{dE_{AB}}{d\tau},$$

the first of Kelvin's relations.

This argument is not affected by the increase of entropy arising from the irreversible Joulean heat, for this is obviously exactly accounted for by the additional energy derived from the iR drop.

To obtain the second relation of Kelvin, the first law alone is sufficient. We notice in the first place, by an argument similar to Kelvin's, that the Peltier heat cannot be the only heating effect. For if this were the only effect, the total e. m. f. of the couple would be accounted for by the difference of Peltier heat at the two junctions, whence at once $\frac{dE}{d\tau} = \frac{dP}{d\tau}$. This, combined with (1), gives $P = a\tau$ and $E = b + a\tau$, where a and b are constants. But these relations are known to be contrary to fact. Hence by the same argument as Kelvin's there must be heating effects in the parts of the metal where there is a temperature gradient, and application of the first law gives at once $\frac{dE_{AB}}{d\tau} = \frac{dP_{AB}}{d\tau} + \sigma_A - \sigma_B$, and this, combined with (1), gives $\tau\frac{d^2E_{AB}}{d\tau^2} = \sigma_B - \sigma_A$, the second Kelvin relation.

It would appear then that by arranging the couple so that the reversible effects result in the maximum possible dissipation of available energy the difficulty in Kelvin's argument may be avoided, and experimental verification of equations (1) and (2) as such becomes superfluous, granting the validity of the first and second laws. This does not mean that the

need for experimental verification has entirely vanished, for in the deduction we assumed as an experimental fact that the Peltier heat is reversible, that is, that on reversing the direction of current through a junction a development of heat becomes a precisely equal absorption. This assumption about the nature of the Peltier heat was also made by Kelvin, and is evidently distinct from the additional assumption contained in his thermodynamic argument. Perhaps the simplest experimental method of showing the reversibility of the Peltier heat would be to show that the e. m. f. of a couple is the same on open and closed circuit. The reversibility of the Thomson heat with direction of current flow does not seem to be necessarily involved in relation (1) and (2), except as E and σ are by implication determined only by temperature and material.

[1] This matter is discussed in greater detail in: P. W. Bridgman, *Phys. Rev.*, **14**, 306–347, 1919.

ON THE NATURE OF THE TRANSVERSE THERMO-MAGNETIC EFFECT AND THE TRANSVERSE THERMO-ELECTRIC EFFECT IN CRYSTALS

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It is known that there is a close formal parallelism between the Ettingshausen temperature gradient set up in an isotropic conductor at right angles to an electric current and a magnetic field, and Kelvin's temperature difference between two opposite sides of a crystal rod, the length of which is oblique to the crystal axis, and in which a longitudinal electric current flows. Not only are these effects geometrically similar, but both reverse sign when the direction of current flow changes, and are, as far as known, proportional to current strength. The parallelism may be carried through for the other effects. The analogue of the Nernst transverse e. m. f. under a longitudinal heat current is the longitudinal e. m. f. in a crystal produced by a transverse temperature difference; the analogue of the Hall effect is the transverse potential gradient in a crystal carrying a longitudinal electric current arising from the fact that the lines of current flow are not perpendicular to the equipotential surfaces, and similarly the analogue of the Righi-LeDuc transverse temperature gradient is the transverse temperature gradient in a crystal carrying a longitudinal heat current arising from the fact that the lines of heat flow are not perpendicular to the surfaces of constant temperature. One

might therefore be tempted to look for an underlying similarity of mechanism in a crystal and an isotropic substance made non-isotropic by a magnetic field; it is the purpose of this note to show that in spite of the close parallelism there is a fundamental difference of sign in the relations connecting the Ettingshausen with the Nernst coefficient and their corresponding crystal analogues, which indicates that the effects must be essentially different in character.

There are two diametrically opposite ways of regarding the Ettingshausen effect. The first of these I presented to the Solvay Congress in 1924, and published later in the *Physical Review*.[1] According to this point of view the Ettingshausen temperature difference between two opposite sides of a plate carrying a longitudinal electric current in a magnetic field is not accompanied by a transverse flow of heat from the hot to the cold side of the plate, but the temperature difference is maintained without heat flow, by what I called a "thermomotive" force, precisely as the two terminals of a battery on open circuit are maintained at a difference of electric potential by the "electro-motive" force of the battery. The difference of temperature between the two sides of the plate may be utilized to operate a thermodynamic engine, allowing heat to flow from the hot side through the engine to the cold side, the return flow being from *cold to hot* in the plate. When such a heat current flows between the sides of the plate, a longitudinal e. m. f. is set up by the Nernst effect (the Nernst effect and the Ettingshausen effect thus appear as one the inverse of the other), and the longitudinal electric current flowing against this Nernst e. m. f. puts into the system the energy extracted by the thermodynamic engine. On equating the energy put in to the energy extracted, the relation $Q = kP/\tau$ is found between the Nernst coefficient Q, the Ettingshausen coefficient P, thermal conductivity k, and absolute temperature τ. This relation is checked by experiment within the accuracy of the measurements.

The second point of view with regard to the Ettingshausen effect is that of Lorentz,[2] also presented to the Solvay Congress. According to this, the Ettingshausen temperature difference is accompanied by a transverse current of heat in the plate by conduction from the hot to the cold side. This point of view may also be made to give a quantitative relation. The argument which I now give is not that employed by Lorentz, but is patterned after one given by me in the preceding paper on the application of thermodynamics to the thermo-electric circuit. The continual conduction of heat in the plate means a continual dissipation of available energy, which means an increase in the entropy of the entire universe, which under these conditions demands a continual rise of the average temperature of the plate. This continual rise of the temperature of the plate must be at the expense of energy put into the system by the current, and the source of

this is obviously the Nernst e. m. f. associated with the transverse heat flow. Working out the quantitative relations gives exactly the same result as before except for a difference of sign, the result now being $Q = -kP/\tau$. The reason for the difference of sign is obvious, for according to the second point of view the current puts energy into the system when the transverse heat current in the plate is from hot to cold, whereas according to the first point of view, energy is put in when the heat current is from cold to hot. Since the negative sign is directly contrary to experiment, the first point of view must be the correct one, and the necessity for the concept of thermo-motive force is established.

Lorentz, in the discussion which he added in the printed Solvay Report several years after the meeting of the Congress, clearly formulated the essential difference between these two points of view, and recognized the necessity of the concept of thermo-motive force. He also commented on the care which must be used in applying the le Chatelier principle to cases like this. Suppose that we apply a transverse temperature difference to opposite sides of a plate in a magnetic field. This temperature difference is accompanied by a transverse heat flow, which gives rise to a longitudinal e. m. f. by the Nernst effect. Now complete the external circuit, allowing a longitudinal electric current to be driven by the Nernst e. m. f. This electric current will produce a transverse temperature difference by the Ettingshausen effect. In what direction is it? It is natural to expect that it must be in such a direction as to diminish the applied temperature difference. But detailed analysis will show at once that such a diminution of the temperature difference would demand the minus sign in the relation between Q and P which is given by the second point of view, and which is contrary to experiment. The Ettingshausen effect is in such a direction that the transverse temperature difference becomes greater. The le Chatelier principle can be maintained only by noting that, in spite of the greater transverse temperature difference, the transverse heat flow has become less, because of the thermomotive force tending to drive heat up the temperature gradient. In other words, in order to maintain the le Chatelier principle, heat flow must be regarded as the fundamental thing and not temperature difference. It is obvious that the le Chatelier principle might easily prove a false guide in fresh fields.

The transverse temperature difference in crystals predicted by Kelvin may now be subjected to an analysis exactly like that above for the Ettingshausen effect. The transverse temperature difference is an experimental fact; the question is whether this temperature difference is maintained without heat flow by a thermo-motive force in the crystal, or whether there is a continual flow of heat and so continual dissipation of energy when a steady current flows obliquely in a crystal. The answer

to this can be given by examining the sign of the inverse effects. If opposite sides of a long crystal bar oblique to the axis are maintained at a difference of temperature, a longitudinal e. m. f. arises, which will drive a current through the bar on completing the external circuit. This current produces in turn a transverse temperature difference—does this oppose or aid the original temperature difference?

The inverse effect, that is, the longitudinal e. m. f. generated by a transverse temperature difference, is very easy to observe, but as far as I am aware, little attention has been paid to the sign of the effect. I have just found with a single crystal of bismuth that the current generated by the inverse effect is in such a direction as to decrease the original temperature difference. That is, the direct and the inverse effect are connected by a relation with the opposite sign from that of the Nernst-Ettingshausen effects. A crystal is not the seat of a thermo-motive force, but if the transverse difference of temperature is allowed to establish itself, there is a continual heat current, and continual dissipation of energy.

The precise connection between the direct and the inverse effect in a crystal may now be found by the counterpart of the argument already indicated. Consider a crystal plate of breadth b and unit depth carrying a longitudinal current of density i (total current ib), and assume for simplicity that the crystal has rotational symmetry about the crystal axis (as do all metals yet investigated) and that this axis lies in the plane of the face b, and makes an angle θ with the length. Furthermore, suppose the bar so long in comparison with the breadth that the Peltier heat at the ends may be neglected in comparison with the transverse heating effects. Consider now a piece of this plate of unit length. Let T denote the transverse generation of heat per unit area of transverse face per unit current density, and let k denote the transverse thermal conductivity. T and k are in general functions of the angle θ, but the precise nature of the functional relation does not concern us here. The difference of temperature between the two sides of the plate is evidently biT/k and the increase of entropy in unit time due to irreversible heat flow transversely is $(biT/k)(iT/\tau^2) = (bi^2T^2)/k\tau^2$. Denote the longitudinal e. m. f, per unit length per unit transverse heat current by e_t. Then the e. m. f. in this case is $e_t iT$, the energy input of the current against this e. m. f. is $e_t i^2 Tb$, the entropy rise per unit time due to this energy input is $e_t i^2 Tb/\tau$, and equating this to the rise of entropy due to irreversible heat conduction gives

$$e_t = \frac{T}{k\tau},$$

the connection between the direct and the inverse effects.

The sign convention with regard to e_t has already been suggested and the positive sign in the relation just deduced has been found to agree with

experiment. A direct (quantitative) experimental verification of the relation is not superfluous, however, as it is conceivable that there might be still some transverse thermomotive action, of less than the full amount. It would be interesting also to find whether the transverse heat current or the transverse temperature gradient is active in the production of the longitudinal e. m. f. This might be tested by finding whether a crystal bar, in which a transverse thermo-motive force is developed by a longitudinal current in a magnetic field, but in which there is no transverse heat flow, is the seat of a longitudinal e. m. f.

It is interesting to carry the analysis a little further. If there is no development of heat within the body of the crystal, the net rise of temperature accounted for by the energy input of the current must be described in thermal terms as due to a difference of the transverse heat at the two faces which are at different temperatures. This gives: $\Delta\tau \frac{d}{d\tau}(iT) = e_t . Ti . ib$, and substituting the values already found for $\Delta\tau$ and e_t gives $\frac{dT}{T} = \frac{d\tau}{\tau}$, or $T = c\tau$, exactly as in the analysis for the ordinary Peltier heat, when the Thomson heat is neglected. We may suspect that this relation will be found not to agree with experiment, and that, therefore, there must be a generation of heat in the body of the crystal when a longitudinal electric current flows across a transverse temperature gradient, that is, a transverse Thomson heat. Denote by σ_t the heat so absorbed per unit time per unit depth per unit temperature difference per unit current density. The equation of energy balance is now

$$\Delta\tau \left[i\sigma_t + \frac{d}{d\tau}(iT) \right] = e_t . Ti . ib,$$

whence

$$\sigma_\tau + \frac{dT}{d\tau} = \frac{T}{\tau}.$$

Expressing T in terms of e_t gives at once

$$\sigma_\tau = -\tau \frac{d(e_t k)}{d\tau}$$

for the transverse Thomson heat.

The precise analogy between these formulas and those for the ordinary thermo-electric circuit is at once obvious, T taking the place of the Peltier heat, σ_t the ordinary Thomson heat, and $e_t k$ the thermoelectric power (that is, the e. m. f. per unit temperature difference) of the couple. The method of argument used here is the same as that in my previous paper on the application of thermodynamics to the thermo-electric circuit, so that the uncertainty in the result of Kelvin arising from neglect of the

irreversible effects is avoided, and we may be sure that these relations are the rigorous result of thermodynamic principles, with no assumptions involving the neglect or irreversible aspects of the phenomena.

The formal thermo-magnetic analogy of the Thomson transverse effect in crystals is an absorption of heat by a heat current flowing transversely in a bar carrying a longitudinal electric current of density i in a perpendicular magnetic field in amount equal to the fraction PHi/T of itself per unit length measured transversely. This may be proved at once from the equation of energy balance.

[1] P. W. Bridgman, *Phys. Rev.*, Dec., 1924, 644–651. Report of the Fourth Solvay Congress, "Conductibilité Électrique des Métaux," 352–354.

[2] H. A. Lorentz, *Fourth Solvay Congress*, 354–360.

ON THE RED SHIFT OF SPECTRAL LINES THROUGH INTERSTELLAR SPACE

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A. *Introduction.*—It is known that very distant nebulae, probably galactic systems like our own, show remarkably high receding velocities whose magnitude increases with the distance. This curious phenomenon promises to provide some important clues for the future development of our cosmological views. It may be of advantage, therefore, to point out some of the principal facts which any cosmological theory will have to account for. Then a brief discussion will be given of different theoretical suggestions related to the above effect. Finally, a new effect of masses upon light will be suggested which is a sort of gravitational analogue of the Compton effect.

B. *Discussion of the Observational Facts.*—(1) E. Hubble[1] has shown recently that the correlation between the apparent velocity of recession and the distance is roughly linear, corresponding to 500 km./sec. per 10^6 parsecs. Large deviations occur for the nearest nebulae, which may be attributed to their peculiar motions. The most recent observations by M. Humason[2] seem to indicate that for very large distances (50×10^6 light years) the individual deviations become so great (3000 km./sec. out of 8000 km./sec.) that they hardly can be due to peculiar motions and must, therefore, be accounted for in some other way.

(2) The relative shift of frequency $\frac{\Delta\nu}{\nu}$ representing the velocity of recession is apparently independent of the frequency. The available range

in the spectrum is not very large, however. Some exceptions have been found, suggesting that $\frac{\Delta\nu}{\nu}$ for H_β is somewhat greater than for H_γ.

(3) No appreciable absorption or scattering of light can be related to the above shift of spectral lines.

(4) The optical image of an extragalactic nebula seems to be as well defined as can be expected from the resolving power of the telescopes. The distance apparently is only geometrically involved and no additional blurring of the images occurs due to some such process as multiple scattering and superposition of incoherent light beams.

(5) The spectral (absorption) lines obtained from these nebulae are not very well defined, but no systematic investigation of their shape has been carried out. According to recent observations by M. Humason the width of the lines ranges between 4 Å and 7 Å for M_{32} and M_{31} the two Andromeda nebulae.

(6) Extrapolating from Hubble's relation to objects in our own galactic system, the velocity of recession would become so small (5 km./sec. for 10,000 parsecs) that it would escape observation. The theoretical considerations proposed by the author in the following made it probable that an appreciable effect should also be observed in our galaxy. This suggestion was tested by Dr. ten Bruggencate, whose work will be published shortly. His essential result is that the velocity of recession of the globular clusters is a function of the galactic latitude, increasing with decreasing latitude.[4]

We proceed now in discussing different theoretical possibilities of accounting for the phenomenon described above.

C. de Sitter's Universe.—It has been pointed out by de Sitter that the special type of a space proposed by himself as representing our universe would imply on the average a velocity of recession of the far distant nebulae. But the linear relation of Hubble's can only be obtained by making some additional assumptions about the distribution of the nebulae. For more detailed information, we refer to a recent paper by R. C. Tolman.[3] Admitting that de Sitter's explanation accounts for the facts listed above in the sections B_1 to B_5, a correlation of the type B_6 for our own galaxy would present an almost unsurmountable obstacle for any theory based on geometry only.

D. *The Compton-Doppler Effect on Free Electrons.*—We know from different sources, that there exist very dilute gaseous masses distributed all over the interstellar spaces. The observations of the steady Ca^+ and Na absorption lines provide one of the most direct proofs of this fact. These observations also show that some of the atoms occur as ions. It may be concluded, therefore, that an adequate number of free electrons be present. One then might expect that the light coming from distant

nebulae would undergo a shift to the red by Compton effect on those free electrons. Now the admissible deflection in one single process is very small, the angular size of the nebulae being indeed less than one degree of arc. For the change in wave-length $\Delta\lambda$ by a single Compton scattering within the above angle, one obtains then $\Delta\lambda \leq 3 \times 10^{-13}$ cm., so that a great number of collisions between the light quanta and the electrons are necessary in order to produce a change $\Delta\lambda \sim 1$ to 100 Å. But then the light scattered in all directions would make the interstellar space intolerably opaque which disposes of the above explanation.

It is possible, of course, that a great number of the electrons possess very high speed. This is suggested by the existence of the cosmic radiation. In this case, an appreciable shift of frequency may be produced by one collision. But still the difficulty of obtaining too much scattered light in all directions can hardly be avoided, at least if use is made of our present knowledge of the intensity distribution due to Compton effect. Also, it is evident that any explanation based on a scattering process like the Compton effect or the Raman effect, etc., will be in a hopeless position regarding the good definition of the images as mentioned under B_4.

E. *The Usual Gravitational Shift of Spectral Lines.*—One might expect a shift of spectral lines due to the difference of the static gravitational potential at different distances from the center of a galaxy. This effect, of course, has no relation to the distance of the observed galaxy from our own system and, therefore, cannot provide any explanation of the phenomenon discussed in this paper. But it might have some bearing on the width of the observed spectral lines as light coming from different points of the distant galaxy will show varying shifts. To get an estimate, we assume the nebulae to be a sphere of the radius $R = 2 \times 10^4$ light years and of a uniform density 10^{-20} gr./cm.$^3 > \rho > 10^{-24}$ gr./cm.^{3} Then we have for the gravitational potential $\Phi(r)$ this relation: $\Delta\Phi = \Phi(R) - \Phi(0) = \frac{2\pi}{3} f\rho R^2$ where $f = 6.68 \times 10^{-8}$ is the universal gravitational constant. And for the above limits of ρ $8 \times 10^{-8} < \frac{\Delta\nu}{\nu} = \frac{\phi}{c^2} < 8 \times 10^{-4}$ (240 km/sec.) where c is the velocity of light. This effect also might cause a violet shift of the light traveling from the outer regions of our galaxy toward the center.

F. *The Gravitational "Drag" of Light.*—According to the relativity theory, a light quantum $h\nu$ has an inertial and a gravitational mass $\frac{h\nu}{c^2}$. It should be expected, therefore, that a quantum $h\nu$ passing a mass M will not only be deflected but it will also transfer momentum and energy to the mass M and make it recoil. During this process, the light quantum will change its energy and, therefore, its frequency. It is hardly possible

to give a completely satisfactory theory of this gravitational analogue of the Compton effect, without making use of the general theory of relativity. But a rough idea of the nature and the magnitude of the effect may be obtained in the following way.

Suppose a mass m to be traveling on a straight line (x-axis) with a uniform velocity v. The mass M is located at the point $P(x,y)$. If v is sufficiently large, then the actual path will differ only little from the straight line. The force F acting between m and M can then be obtained in the first approximation by assuming that m is traveling along the x-axis with the constant initial velocity v. If m for $t = 0$ at $x = 0$, then

$$F_x = -\frac{\partial \Phi}{\partial x} M \qquad \Phi(x,y,t) = -fm/\sqrt{y^2 + (x - vt)^2}$$

(Φ = gravitational potential at P.)

The x component of the momentum ΔG transferred to M during the time T is

$$\Delta G_x = +\int_0^T F_x dt = -M \int_0^T \frac{\partial \Phi}{\partial x} dt$$

$$= \frac{M}{v} \int_0^T \frac{\partial \Phi}{\partial t} dt = \frac{M}{v} [\Phi(T) - \Phi(0)].$$

If from $t = o$ to $t = T$ the particle has traveled the distance $L = vT$, then

$$\Delta G_x = -\frac{fmM}{v}\left[\frac{1}{\sqrt{(L-x)^2 + y^2}} - \frac{1}{\sqrt{x^2 + y^2}}\right] = \frac{fmM}{v} g_o(x,y).$$

Suppose that matter is distributed all over space with a uniform density ρ and put $M = \rho . 2\pi\, ydydx$, then the x-component of the total momentum lost by m will be

$$G_x = \lim_{D=\infty} \frac{fm}{v} \int_{-D}^{D+L} \int_0^D 2\pi y\rho\, g_o(x,y)\, dx\, dy = 0$$

An exchange of momentum in the x-direction results in this case only if we consider the actually occurring deflection of m from the straight path. This is a second order effect.

In the above calculation, we have assumed that the gravitational interaction is transmitted instantaneously. Let us consider now the case when gravity waves travel with the velocity of light c. Then we have according to the theory of the retarded potentials,

$$\Phi(x,y,t) = -\left.\frac{fm}{r(1 - v_r/c)}\right|_{t'=t-r/c}$$

where the expression on the right has to be taken at the time of emission t' of an action reaching P at the time t. For the distance r of m and M at t' we have,

$$r^2 = y^2 + [x - v(t - r/c)]^2$$

and

$$v_r = [x - v(t - r/c)]v/r.$$

We notice again

$$\frac{\partial \Phi}{\partial x} = -\frac{1}{v}\frac{\partial \Phi}{\partial t}$$

and

$$\Delta G_x = -M\int_{t_0}^{t_T}\frac{\partial \Phi}{\partial x}\,dt = \frac{M}{v}\int_{t_0}^{t_T}\frac{\partial \Phi}{\partial t}\,dt = \frac{M}{v}[\Phi(t_T) - \Phi(t_0)].$$

The disturbance caused by the motion of m from $x = 0$, $t = 0$ to $t = T$, $x = L = vT$ is acting on M from $t_0 = \sqrt{x^2 + y^2}/c$ to $t_T = T + 1/c\sqrt{(L - x)^2 + y^2}$. We obtain therefore,

$$\Delta G_x = -\frac{fmM}{v}\left[\frac{1}{\sqrt{(L - x)^2 + y^2} + (L - x)v/c} - \frac{1}{\sqrt{x^2 + y^2} - xv/c}\right].$$

Developing for $v/c \ll 1$

$$\Delta G_x = \frac{fmM}{v}[g_0(x,y) + g_1(x,y)\,v/c + \ldots]$$

$g_0(x,y)$ is the same function as above

$$g_1(x,y) = \frac{L - x}{(L - x)^2 + y^2} + \frac{x}{x^2 + y^2}.$$

Assuming again a uniform distribution of matter in space, we put $M = 2\pi\, ydydx, \rho$. The integration over g_0 gives zero as before and (for $D \gg L$)

$$2\pi\int_{-D}^{D+L}\int_0^D g_1(x,y)ydxdy = 2\pi lg2LD.$$

ΔG_x corresponding to the matter in a region $-D < x < D + L$ and $0 < y < D$ is therefore

$$\Delta G_x = 1.4\pi\, fm\rho LD/c.$$

In regard to D, it must be remarked that it should be as large as the dimension of the space over which masses are distributed, if those masses are regarded as independent from each other. But the masses are in reality coupled by gravitational forces and the effect of an external perturbation upon them must be computed by considering the system of the far distant masses as a whole. The correct theory will probably have to

be worked out in terms of absorption of gravitational waves. But I think it may be safely assumed that the distance D in which the perturbing effect of the moving mass m begins to fade out is very large compared with the mutual distances of the single masses M in which matter is essentially concentrated.

Going over to the case of light, we have $v = c$ and $m = h\nu/c^2$. We conclude by analogy that a relation of the above type still is valid, especially as it can be derived by simply using dimensional reasoning. Light traveling a distance L then would lose the momentum

$$\Delta\left(\frac{h\nu}{c}\right) = \frac{1.4\pi f\rho DL}{c} \times \frac{h\nu}{c^2} \text{ and } \frac{\Delta\nu}{\nu} = 1.4\pi f\rho DL/c^2.$$

Let us compare this result with the observations.

For the total space investigated, the possible limits for ρ are according to E. Hubble 10^{-26} gr./cm.$^3 > \rho > 10^{-31}$. The mutual distance l of the galactic systems being of the order $l = 10^6$ parsecs, we may assume D for instance of the order 1000 $l = 3 \times 10^{27}$ cm. Then $\Delta\nu/\nu$ for $L = 10^6$ parsecs according to our formula will be in the limits $3 \times 10^{-2} > \Delta\nu/\nu > 3 \times 10^{-7}$. From Hubble's linear relation, we have $\Delta\nu/\nu \sim 1/600$ for the same L. In view of this agreement in order of magnitude, a further elaboration of the theory seems to be worthwhile.

Applying the above theory to globular clusters of our own galaxy it would be essential to take into account the actual mass distribution. We will, however, obtain an estimate of the order of magnitude by taking $L = 15000$ parsecs, $D = 1000\,l$ with $l = 1$ parsec for the mutual average distance of the stars. The limits for the density are 10^{-20} gr./cm.$^3 > \rho_g > 10^{-24}$ gr./cm. The redshift therefore would be $4.2 \times 10^{-4} > \frac{\Delta\nu}{\nu} > 4.2 \times 10^{-8}$.

Dr. ten Bruggencate has, in fact, been able to establish a relation between the redshift and the distribution of matter in space. He finds $\frac{\Delta\nu}{\nu} \sim {}^1/_{1000}$ for light traveling through a distance of 15,000 parsecs in the galactic plane. It would be very important to measure the radial velocities of as many globular clusters as possible in order to decide definitely between the different theories. It is especially desirable to determine the redshift independent of the proper velocities of the objects observed. This might, for instance, be done with help of the steady calcium lines. It is easy to see that the above redshift should broaden these absorption lines asymmetrically toward the red. If these lines can be photographed with a high enough dispersion, the displacement of the center of gravity of the line will give the redshift independent of the velocity of the system from which the light is emitted.

The explanation of the apparent velocity of recession of distant nebulae proposed in this paper is in qualitative accordance with all of the observational facts known so far. It is therefore desirable, in the first place, to place the computations on a sound theoretical basis involving the general theory of relativity. In the second place, the transfer of momentum from the light to the surrounding masses should be determined taking into account all of the mutual gravitational interactions. Thirdly, it is evident that the proper motions of these masses will play some rôle. Shifts of the spectral lines to the violet should indeed be expected for thermodynamic reasons if light is traveling through systems of masses with very high average velocities. Finally, it might be interesting to study the gravitational drag exerted by light upon light.

I wish to thank Dr. ten Bruggencate who kindly set out on the difficult task of testing some of the suggestions presented in this paper.

[1] E. Hubble, *Proc. Nat. Acad. Sci.*, 1929, **15**, 168.

[2] I am indebted to Mr. M. Humason and to Dr. E. Hubble for private information.

[3] R. C. Tolman, *Astrophys. J.*, **49**, 245, 1929.

[4] A paper by E. von der Pahlen and E. Freundlich in the *Publikationen des Astrophysikalischen Observatoriums zu Potsdam*, **86**, Bd. 26, Heft 3, should be mentioned in this connection. Fig. 6 on page 44 of this paper also shows a correlation between the radial velocity of globular clusters and the galactic latitude. The authors, however, interpret it as being due to real motions.

RELATIVE INTENSITIES IN NUCLEAR SPIN MULTIPLETS

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Introduction.—The discovery and resolution in certain of the atomic lines of Bi, Cs, and Tl, of a fine structure[1,2,3] much smaller than that ordinarily attributed to the electron spin has led to the notion that the nucleus of an atom may be possessed of a spin moment which is capable of interacting with the outer electrons. The order of magnitude of this interaction is very small, as the magnetic moment associated with the nuclear spin is much smaller than the corresponding moment associated with the electron spin. Very recently a paper has appeared by Hargreaves[4] in which the consequences of this picture are worked out in detail for the particular case of a single electron in the Coulomb field of a nucleus which is also possessed of a spin moment $\frac{1}{2}(h/2\pi)$. The method used is that of Pauli with multiple wave functions, the effect of the nuclear spin being regarded as a small perturbation of the multiplet states which are

due to the electron spin. As this method requires the use of $(2i + 1)(2^f)$ simultaneous wave functions, where i is the nuclear spin quantum number and f is the number of electrons, the computational difficulties are prohibitive for an arbitrary value of i, even for the case of a single electron. However, by the use of previously known expressions for the coördinate matrices it is possible to give intensity formulae applicable in the general case, provided one makes suitable assumptions regarding the type of quantization introduced by the nuclear spin.

Notation.—We shall use a notation which in the main is designed to be in accord with that suggested for atomic spectra.[5] All angular momenta are measured in units $h/2\pi$. The resultant electronic orbital angular momentum will be designated by L, the resultant spin by S and the total angular momentum (vector resultant of L and S) by J. The nuclear spin will be designated by i, whose value will be left indeterminate except as may be specified in certain sections of this paper. In an external magnetic field strong enough to overpower the interactions between i and the rest of the atom, the vectors i and J are quantized independently with respect to the axis of the external field, their projections on this axis being measured by the usual quantum numbers M_i and M.

Theory.—We start with our unperturbed system as an atom in a magnetic field sufficiently strong to overpower the interaction of i with the electronic structure, but which is still so small that it does not produce Zeeman patterns of appreciable magnitude compared to the J multiplets. We have, then, the system of quantization mentioned above in which the multiplets are characterized by the quantum numbers (L, S, J, i, M_i, M).

To get the system of quantization which is applicable in the absence of the magnetic field, one should know the form of the perturbative Hamiltonian between the nuclear spin and the outer electrons. Hargreaves neglects the interaction energy between the nuclear and electron spins, and finds that the remaining coupling terms are approximately given by the usual "cosine" law between i and L. The neglect of the interactions between the two types of spin does not seem to the present writer to be any too justifiable, as according to the usual notions of the nature of spins, the coupling between the spin of one electron and the orbital angular momentum of a second is of the same order of magnitude as the coupling between the spins.[6] A similar result might be expected to obtain in Dirac's relativistic theory of the spin, but this is only a guess.

This question is, however, a matter of secondary importance for most purposes of this paper, as in any case it seems quite reasonable to suppose that if we include only terms in the perturbative Hamiltonian which do not disappear on averaging over the precession around J (i.e., keep only elements diagonal in J), then the interaction will be such that in the absence of an external field the vectors i and J will be quantized to a re-

sultant f, such that[7] $f = |J - i|, |J - i| + 1, \ldots, J + i$. This is the type of quantization which would result from assuming any "cosine" law whether between i and L alone as is done by Hargreaves, or between i and L and also between i and S, even with different coupling constants; it is, in fact, the usual method of coupling vectors and has already been used by Back and Goudsmit[1] and by Jackson[2] in discussing their experimental results.

The projection of f on an external axis, which we may identify with the axis of the magnetic field, is also quantized with the quantum number M_f, where $|M_f| \leqq f$. This quantum number is of course an invariant between the zero and strong field types of quantization discussed above, and has the value $M_f = M_i + M$.

Intensities in Strong Fields.—The intensity relations in case i and J precess independently about the external axis are precisely the same as those which are applicable when the nuclear spin is left out of account, and have been given by Kronig[8] and others.[9]

Intensities in Zero Fields.—The intensity formulae in zero fields can be readily obtained from those previously obtained by Kronig.[8] To do this we have only to note that according to our assumption as to the type of quantization which is applicable, the coupling of J and i to give f is quite the same as that of L and S to the resultant J. Or, in Kronig's notation, we see that S, L, J on the one hand and i, J, f on the other should enter in the formulae in an analogous manner to his R, K, J (which are $^1/_2$ unit higher than our quantum numbers, as Kronig used the old Landé notation). Thus in Kronig's formulae we have only to introduce another factor of the form of his F's in order to take care of the precession about f. This procedure does not assure one that the formulae so obtained will satisfy the principle of spectroscopic stability as they must in order to be in accord with the new quantum mechanics.[10] If we take $|q(n;n')|^2 = |x(n;n')|^2 + |y(n;n')|^2 + |z(n;n')|^2$, so that $|q(n;n')|^2$ is proportional to the total intensity radiated in the transition $n \longrightarrow n'$, the principle of spectroscopic stability shows that an expression such as

$$\sum_{M_i,M_i'}^{M,M'} \left| q(a, L, J, M_i, M; a', L', J', M_i', M') \right|^2$$

is invariant of the mode of quantization and so must be equal to

$$\sum_{f,f'}^{M_f,M_f'} \left| q(a, L, J, f, M_f; a', L', J', f', M_f') \right|^2.$$

In the present case this is simply the requirement that the total intensity radiated in a given transition $(a, L, J) \longrightarrow (a', L', J')$ shall remain unchanged by the introduction of the nuclear spin, except, of course, for the appearance of the new *a priori* probability factor $(2i + 1)$.

The following expressions will be found to satisfy these requirements. As their application is of importance only in case the external field is zero, we shall give them already summed over the Zeeman components associated with the quantum number M_f.

$$\Delta L = +1, \quad \Delta J = +1,$$

$$|\, q(f; f+1)\,|^2 = D(L; L+1)\left[\frac{E(J+1; L; S)E(J+2; L; S)}{4U(J+1)} \times \frac{E(f+1; J; i)E(f+2; J; i)}{f+1}\right]$$

$$|\, q(f; f)\,|^2 = D(L; L+1)\left[\frac{E(J+1; L; S)E(J+2; L; S)}{4U(J+1)} \times (2f+1)\,\frac{E(f+1; J; i)\,F(f-1; J; i)}{f(f+1)}\right]$$

$$|\, q(f; f-1)\,|^2 = D(L; L+1)\left[\frac{E(J+1;\ L;\ S)E(J+2;\ L;\ S)}{4U(J+1)} \times \frac{F(f-1; J; i)F(f-2; J; i)}{f}\right]$$

$$\Delta L = +1, \quad \Delta J = 0,$$

$$|\, q(f; f+1)\,|^2 = D(L; L+1)\left[\frac{E(J+1;\ L;\ S)F(J-1;\ L;\ S)}{4V(J)} \times \frac{E(f+1; J; i)\,F(f; J; i)}{f+1}\right]$$

$$|q(f; f)\,|^2 = D(L; L+1)\left[\frac{E(J+1; L; S)\,F(J-1; L; S)}{4V(J)} \times (2f+1)\,\frac{\{G(f; J; i)\}^2}{f(f+1)}\right]$$

$$|\, q(f; f-1)\,|^2 = D(L; L+1)\left[\frac{E(J+1;\ L;\ S)F(J-1;\ L;\ S)}{4V(J)} \times \frac{E(f;\ J;\ i)F(f-1; J; i)}{f}\right]$$

$$\Delta L = +1, \quad \Delta J = -1,$$

$$|\, q(f;f+1)\,|^2 = D(L;L+1)\left[\frac{F(J-1;\,L;\,S)F(J-2;\,L;\,S)}{4U(J)} \times \frac{F(f+1;\,J;\,i)\,F(f;\,J;\,i)}{f+1}\right]$$

$$|\, q(f;f)\,|^2 = D(L;L+1)\left[\frac{F(J-1;\,L;\,S)F(J-2;\,L;\,S)}{4U(J)} \times (2f+1)\,\frac{E(f;\,J;\,i)F(f;\,J;\,i)}{f(f+1)}\right]$$

$$|q(f;f-1)\,|^2 = D(L;L+1)\left[\frac{F(J-1;\,L;\,S)F(J-2;\,L;\,S)}{4U(J)} \times \frac{E(f;\,J;\,i)E(f-1;\,J;\,i)}{f}\right].$$

where
$$E(a;\,b;\,c) = (a+b)(a+b+1) - c(c+1),$$
$$F(a;\,b;\,c) = c(c+1) - (a-b)(a-b+1),$$
$$G(a;\,b;\,c) = a(a+1) + b(b+1) - c(c+1),$$
$$U(a) = a^2(2a-1)(2a+1), \quad V(a) = a^2(a+1)^2.$$

$D(L;\,L+1)$ is a constant which is proportional to the total intensity radiated in the transition $(a,L) \longrightarrow (a',\,L+1)$, where a and a' are the electronic quantum numbers associated with the initial and final states. We do not write a and a' in the elements of D and q as should be done for completeness, but their presence is naturally to be implied throughout. Finally, $D(a,\,L;\,a',\,L') = D(a',\,L';\,a,\,L)$.

The intensities for a transition in which $\Delta L = -1$ can be obtained from the above by using the Hermitian property; i.e., $|\, q(L,J,f;L',J',f')\,|^2 = |\, q(L',J',f';\,L,J,f)\,|^2$.

These expressions have the general property that the total intensity radiated in a transition $(L,J,f) \longrightarrow (L',J')$ is proportional to $(2f+1)$, and similarly the total intensity radiated in a transition $(L,J) \longrightarrow (L')$ is proportional to $(2J+1)$. Also as the state $f = 0$ can only exist when $J = i$, it can easily be verified that there is no intensity radiated in a transition for which $f = 0 \longrightarrow f = 0$.

As a check on these formulae, we first compare them with the results of Hargreaves[4] for the special case $i = {}^1/_2$, $S = {}^1/_2$, as embodied in his figure 3. If we choose our undetermined constant $D(L;\,L+1)$ such that it is ${}^1/_4$ for $L = 0 \longrightarrow L = +1$, and is 2 for $L = 1 \longrightarrow L = 2$, our results are identical with his.

A more interesting thing is to try these formulae on the line λ4722 Å of Bi for which $i = 9/2$, $S = {}^1/_2$, as measured by Back and Goudsmit.[1] Referring to the table of intensities given with their figure 2, and normal-

izing the results so that the component λ4722.333 has intensity 10, our formulae predict the relative intensities (5.4; 5.1; 3.4)(1.8; 5.1; 10) where Back and Goudsmit give as experimental values (8; 7; 4)(1; 8; 10). The writer does not know whether the discrepancies between these values would lie outside the probable error in the experimental intensities.

It may be noted that there is a certain formal similarity between the calculation of this paper and that of Houston[11] on coupling in two electron systems. The spin field of his s electron gives an effect similar to that which we have assumed for the nucleus.

The author is greatly indebted to Professor J. H. Van Vleck for valuable hints, and also for the reference to Houston's paper.

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[1] E. Back and S. Goudsmit, *Zeits. Physik*, **47**, 174, 1928.

[2] D. A. Jackson, *Proc. Roy. Soc.* **121A**, 432, 1928.

[3] H. Schüler and H. Brück, *Zeits. Physik*, **55**, 575, 1929.

[4] J. Hargreaves, *Proc. Roy. Soc.*, **124A**, 568, 1929.

[5] Russell, Shenstone and Turner, *Phys. Rev.*, **33**, 900, 1929.

[6] Cf. Heisenberg, *Zeits. Physik*, **39**, 499, 1926.

[7] It is to be noted that some of the conclusions of Schüler and Brück do not agree with ours because of the different methods of compounding vectors; e.g., for a $^2D_{3/2}$ term with $i = 1/2$ they predict a triplet level where we predict a doublet.

[8] R. de L. Kronig, *Zeits. Physik*, **31**, 885, 1925. Kronig's expressions have been justified in the quantum mechanics by Dirac, *Proc. Roy. Soc.*, **111A**, 281, 1926.

[9] A. Sommerfeld and H. Hönl, *Sitzungsberichte der Preuss. Akad. der Wissen.*, **9**, 141, 1925.

[10] Cf. E. L. Hill and J. H. Van Vleck, *Phys. Rev.*, **32**, 250, 1928, for a similar application of spectroscopic stability.

[11] W. V. Houston, *Phys. Rev.*, **33**, 297, 1929.

ON EINSTEIN'S UNIFIED FIELD EQUATIONS AND THE SCHWARZSCHILD SOLUTION

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In recent papers[1] Wiener and the author have determined the tensors ${}^5h_\lambda$ of Einstein's unified theory of electricity and gravitation under the assumptions of static spherical symmetry and of symmetry of past and future. It was there shown that the field equations suggested in Einstein's second 1928 paper[2] lead in this case to a vanishing gravitational field. The purpose of this paper is to investigate, for the same case, the nature of the gravitational field obtained from the field equations suggested by Einstein in his first 1929 paper.[3] These field equations are

$$[h(\varphi_{\underline{\alpha};\,\underline{\beta}} - \varphi_{\underline{\beta};\,\underline{\alpha}})]_{/\alpha} = 0 \tag{1}$$

$$\overline{V}^{\alpha}_{\underline{\beta\gamma}/\gamma} - \overline{V}^{\sigma}_{\underline{\beta\nu}}\Lambda^{\alpha}_{\sigma\nu} = 0. \tag{2}$$

A word of explanation of the symbolism found here is perhaps necessary. The tensors appearing in the formulas are defined as follows:

$$\Lambda^{\mu}_{\nu\sigma} = \Gamma^{\mu}_{\nu\sigma} - \Gamma^{\mu}_{\sigma\nu}, \qquad \Gamma^{\mu}_{\nu\sigma} = {}_{s}h^{\mu}\frac{\partial {}^{s}h_{\nu}}{\partial x^{\sigma}},$$

$$\varphi_{\alpha} = \Lambda^{\mu}_{\alpha\mu}, \qquad h = \left| {}^{s}h_{\lambda} \right|,$$

$$V^{\alpha}_{\mu\nu} = \Lambda^{\alpha}_{\mu\nu} + \varphi_{\nu}\delta^{\alpha}_{\mu} - \varphi_{\mu}\delta^{\alpha}_{\nu}.$$

δ^{μ}_{ν} is Kronecker's symbol; a semicolon indicates affine covariant differentiation corresponding to $\Gamma^{\mu}_{\nu\sigma}$, and a bar over a tensor indicates the corresponding tensor density (e.g., $\overline{T}\,\ldots = h\,T\,\ldots$). An underlined index indicates that the index in question is to be raised (or lowered), and the symbol $/\alpha$ denotes the operation closely akin to divergence

$$\overline{T}\,\ldots{}_{/\sigma}^{\sigma} = hT\,\ldots{}_{;\,\sigma}^{\sigma} - \overline{T}\,\ldots{}^{\alpha}\,\Lambda^{\sigma}_{\alpha\sigma}. \tag{3}$$

The expression $\varphi_{\mu;\,\alpha} - \varphi_{\alpha;\,\mu}$, which according to Einstein defines the electromagnetic field tensor, has the value

$$E_{\mu\alpha} = \varphi_{\mu;\,\alpha} - \varphi_{\alpha;\,\mu} = \frac{\partial\varphi_{\mu}}{\partial x^{\alpha}} - \frac{\partial\varphi_{\alpha}}{\partial x^{\mu}} + \Lambda^{\sigma}_{\alpha\mu}\varphi_{\sigma} = F_{\mu\alpha} + \Lambda^{\sigma}_{\alpha\mu}\varphi_{\sigma},$$

$F_{\mu\alpha}$ is the ordinary electromagnetic tensor. Equation (1) may therefore be written

$$\overline{E}_{\underline{\mu\alpha}/\alpha} = 0. \tag{1a}$$

We also have

$$h\,V^{\alpha}_{\underline{\beta\gamma};\,\gamma} = \overline{V}^{\alpha}_{\underline{\beta\gamma};\,\gamma} = h\frac{\partial V^{\alpha\beta\gamma}}{\partial x^{\gamma}} + \Gamma^{\alpha}_{\sigma\gamma}\overline{V}^{\sigma\beta\gamma} + \Gamma^{\beta}_{\sigma\gamma}\overline{V}^{\alpha\sigma\gamma} + \Gamma^{\gamma}_{\sigma\gamma}\overline{V}^{\alpha\beta\sigma};$$

therefore, from (3), Equation (2) may be written

$$h\frac{\partial V^{\alpha\beta\gamma}}{\partial x^{\gamma}} + \overline{V}^{\sigma\beta\gamma}\Gamma^{\alpha}_{\sigma\gamma} + \overline{V}^{\alpha\sigma\gamma}\Gamma^{\beta}_{\sigma\gamma} + \overline{V}^{\alpha\beta\sigma}\Gamma^{\gamma}_{\sigma\gamma} - \overline{V}^{\alpha\beta\sigma}\Lambda^{\gamma}_{\sigma\gamma} - \overline{V}^{\sigma\beta\gamma}\Lambda^{\alpha}_{\sigma\gamma} = 0;$$

or simplifying and dividing through by h

$$\frac{\partial V^{\alpha\beta\gamma}}{\partial x^{\gamma}} + V^{\sigma\beta\gamma}\Gamma^{\alpha}_{\gamma\sigma} + V^{\alpha\sigma\gamma}\Gamma^{\beta}_{\sigma\gamma} + V^{\alpha\beta\gamma}\Gamma^{\gamma}_{\gamma\sigma} = 0. \tag{2a}$$

In accordance with the assumptions of static spherical symmetry and symmetry of past and future the line element is taken to be $ds^2 = U^2(r)dr^2 + r^2\sin^2\theta d\varphi^2 + r^2d\theta^2 + W^2(r)dt^2$. Using Cartesian coordinates in the local quadruples and Cartesian (Gaussian) coördinates

in the space-time continuum, it is readily found by availing ourselves of the results of our second paper (loc. cit.) that there are 18 non-vanishing components of the tensor $\Lambda^{\mu}_{\nu\sigma}$ as follows:

$$\begin{aligned}
\Lambda^1_{12} &= -\Lambda^1_{21} = -\Lambda^3_{23} = \Lambda^3_{32} = -Ay \\
\Lambda^1_{13} &= -\Lambda^1_{31} = -\Lambda^2_{32} = \Lambda^2_{23} = -Az \\
\Lambda^2_{12} &= -\Lambda^2_{21} = \Lambda^3_{13} = -\Lambda^3_{31} = Ax \\
\Lambda^4_{14} &= -\Lambda^4_{41} = -xW'/Wr \\
\Lambda^4_{24} &= -\Lambda^4_{42} = -yW'/Wr \\
\Lambda^4_{34} &= -\Lambda^4_{43} = -zW'/Wr
\end{aligned}$$

with the abbreviations $A = (u - 1)/r^2$ and $W' = \partial W/\partial r$. The components of the electromagnetic potential φ_μ are, therefore, as already noted before [loc. cit. (1), second paper]

$$\varphi_1 = 2Ax, \quad \varphi_2 = 2Ay, \quad \varphi_3 = 2Az, \quad \varphi_4 = 0.$$

The non-zero components of the tensor $V^{\alpha}_{\beta\gamma}$ are

$$\begin{aligned}
V^1_{12} &= -V^1_{21} = -V^3_{23} = V^3_{32} = Ay \\
V^2_{12} &= V^3_{13} = -V^2_{21} = -V^3_{31} = -Ax \\
V^1_{13} &= V^2_{23} = -V^2_{32} = -V^1_{31} = Az \\
V^4_{14} &= -V^4_{41} = -x\left(\frac{W'}{Wr} + 2A\right) \\
V^4_{24} &= -V^4_{42} = -y\left(\frac{W'}{Wr} + 2A\right) \\
V^4_{34} &= -V^4_{43} = -z\left(\frac{W'}{Wr} + 2A\right),
\end{aligned}$$

and of the contravariant tensor $V^{\alpha\beta\gamma}$

$$\begin{aligned}
V^{112} &= -V^{121} = -V^{323} = V^{332} = Ay/U^2 \\
V^{113} &= -V^{131} = V^{223} = -V^{232} = Az/U^2 \\
V^{212} &= -V^{221} = V^{313} = -V^{331} = -Ax/U^2 \\
V^{414} &= -V^{441} = -\frac{x}{U^2W^2}\left(\frac{W'}{Wr} + 2A\right) \\
V^{424} &= -V^{442} = -\frac{y}{U^2W^2}\left(\frac{W'}{Wr} + 2A\right) \\
V^{434} &= -V^{443} = -\frac{z}{U^2W^2}\left(\frac{W'}{Wr} + 2A\right).
\end{aligned}$$

We now consider the "electromagnetic" equation (1), or (1a). We obtain in this case for all possible combinations of μ and α

$$E_{\mu\alpha} = 0.$$

Equation (1) reduces, therefore, to a trivial identity. The vanishing of the electrostatic field, already emphasized in our previous papers, is of

course a consequence of the definition of the electromagnetic potential and is independent of the particular choice of the field equations.

We now turn our attention to the "gravitational" equations (2a). The free indices are here α and β, and we have the following five possible types of equations: $\alpha = 1, \beta = 1$; $\alpha = 1, \beta = 2$; $\alpha = 1, \beta = 4$; $\alpha = 4, \beta = 1$; $\alpha = 4, \beta = 4$. Of these, two reduce to trivial identities, i.e., for $\alpha = 1, \beta = 4$ and for $\alpha = 4, \beta = 1$. The equation corresponding to $\alpha = 1, \beta = 1$ is

$$\frac{\partial V^{112}}{\partial y} + \frac{\partial V^{113}}{\partial z} + (2\Gamma^1_{12} + \Gamma^2_{22} + \Gamma^3_{32} + \Gamma^4_{42})V^{112} + (2\Gamma^1_{13} + \Gamma^2_{23} + \Gamma^3_{33} + \Gamma^4_{43})V^{113} + \Gamma^1_{22}V^{212} + \Gamma^1_{33}V^{313} = 0,$$

which on substituting the values of the different quantities and performing the indicated operations yields

$$\frac{2}{Ur^2} - \frac{2}{U^2r^2} + \frac{y^2+z^2}{U^2}\left(\frac{U'}{r^3} - \frac{2U}{r^4} + \frac{2}{r^4}\right) - \frac{2(y^2+z^2)U'(U-1)}{U^3r^3} + \frac{U-1}{r^2U}(y^2+z^2)\left(\frac{U'}{Ur} + \frac{W'}{Wr}\right) - \frac{2(U-1)}{r^2U^2}\left(\frac{1}{r^2} - \frac{1}{Ur^2}\right)(r^2 - y^2 - z^2) = 0.$$

Likewise for $\alpha = 2, \beta = 2$ and for $\alpha = 3, \beta = 3$ we obtain the analogous equations

$$\frac{2}{Ur^2} - \frac{2}{U^2r^2} + \frac{x^2+z^2}{U^2}\left(\frac{U'}{r^3} - \frac{2U}{r^4} + \frac{2}{r^4}\right) - \frac{2(x^2+z^2)U'(U-1)}{U^3r^3} + \frac{U-1}{r^2U}(x^2+z^2)\left(\frac{U'}{Ur} + \frac{W'}{Wr}\right) - \frac{2(U-1)}{r^2U^2}\left(\frac{1}{r^2} - \frac{1}{Ur^2}\right)(r^2 - x^2 - z^2) = 0.$$

$$\frac{2}{Ur^2} - \frac{2}{U^2r^2} + \frac{x^2+y^2}{U^2}\left(\frac{U'}{r^3} - \frac{2U}{r^4} + \frac{2}{r^4}\right) - \frac{2(x^2+y^2)U'(U-1)}{U^3r^3} + \frac{U-1}{r^2U}(x^2+y^2)\left(\frac{U'}{Ur} + \frac{W'}{Wr}\right) - \frac{2(U-1)}{r^2U^2}\left(\frac{1}{r^2} - \frac{1}{Ur^2}\right)(r^2 - x^2 - z^2) = 0.$$

These three equations reduce to the following pair

$$\frac{1}{U} = 1$$

$$\frac{1}{U}\left(\frac{U'}{r} - \frac{2U}{r^2} + \frac{2}{r^2}\right) - \frac{2U'(U-1)}{U^2r} + (U-1)\left(\frac{U'}{Ur} + \frac{W'}{Wr}\right) + \frac{2(U-1)}{U}\left(\frac{1}{r^2} - \frac{1}{Ur^2}\right) = 0.$$

the only solution of which is $U = 1$.

The equation belonging to the type corresponding to $\alpha = 1$, $\beta = 2$ is

$$\frac{\partial V^{121}}{\partial x} + V^{112}\Lambda^2_{12} + (2\Gamma^1_{11} + \Gamma^2_{21} + \Gamma^3_{31} + \Gamma^4_{41})V^{121} + \Gamma^1_{12}V^{221} + \Gamma^1_{32}V^{223} + \Gamma^1_{33}V^{323} = 0,$$

which gives

$$-\frac{U'}{U} - \frac{UW'}{W} + \frac{W'}{W} - \frac{U}{r} - \frac{2}{Ur} + \frac{3}{r} = 0.$$

The solution obtained above is immediately seen to be consistent with this equation.

The equation belonging to the type $\alpha = 4$, $\beta = 4$ is

$$\frac{\partial V^{441}}{\partial x} + \frac{\partial V^{442}}{\partial y} + \frac{\partial V^{443}}{\partial z} + \Gamma^4_{41}V^{441} + \Gamma^4_{42}V^{442} + \Gamma^4_{43}V^{443} + (\Gamma^1_{11} + \Gamma^2_{21} + \Gamma^3_{31} + \Gamma^4_{41})V^{441} + (\Gamma^1_{12} + \Gamma^2_{22} + \Gamma^3_{32} + \Gamma^4_{42})V^{442} + (\Gamma^1_{13} + \Gamma^2_{23} + \Gamma^3_{33} + \Gamma^4_{43})V^{443} = 0,$$

which reduces to

$$W'' + \frac{2W'}{r} - \frac{W'^2}{W} = 0,$$

which is not the Schwarzschild equation for W.[4] It is therefore seen that the field equations suggested by Einstein in his first paper of 1929 do not yield the Schwarzschild solution. While space-time is not flat, space alone is Euclidean.

It may be mentioned in closing that Wiener has shown in a paper to be published elsewhere soon that the Schwarzschild solution satisfies *exactly* the field equations suggested by Einstein in his second 1929 paper.[5] This Wiener proves using precisely the same tensors of this paper. Indeed, as he has shown, these latest field equations reduce identically to the equations of the gravitational theory of 1916.

[1] N. Wiener and M. S. Vallarta, "On the Spherically Symmetrical Statical Field in Einsten's Unified Theory of Electricity and Gravity," these PROCEEDINGS, **15**, 353, 1919; *Ibid.*, "On the Spherically Symmetrical Statical Field in Einstein's Unified Theory: A Correction," these PROCEEDINGS **15**, p. 802.

[2] A. Einstein, "Neue Möglichkeit für eine einheitliche Feldtheorie von Gravitation und Elektrizität," *Sitzungsber. preuss. Akad. Wiss.*, **1928**, pp. 224–227.

[3] A. Einstein, "Zur einheitlichen Feldtheorie," *Ibid.*, **1929**, pp. 2–7.

[4] See for example A. S. Eddington's *Mathematical Theory of Relativity*, p. 85, Cambridge, **1923**.

[5] A. Einstein, "Einheitliche Feldtheorie und Hamiltonsches Prinzip," *Sitzungsber. preuss. Akad. Wiss.*, **1929**, pp. 156–159.

ELEMENTS IN THE SUN

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The identification of elements common to the sun, stars and nebulae bears upon the question of the uniformity of the material constitution of nature. In the sun we have a special interest because of its importance to us and of the possibility of applying the results to specific solar and stellar problems.

For the *Revision of Rowland's Preliminary Table of Solar Spectrum Wave-Lengths*[1] recently completed at Mount Wilson Observatory the original identifications of Rowland were checked and corrected in the light of spectroscopic data that have become available since Rowland's time. Many additional solar lines have been identified with terrestrial elements, and a number of elements not previously recognized in the sun have been added to the original list.

A. BASIS OF IDENTIFICATION. 1. *Coincidences.*—The original identifications of the Fraunhofer lines in Rowland's Table depend upon the agreement in position of solar and laboratory spectral lines—the method of "coincidence." The agreement in wave-length remains, in general, the first line of attack, but considerations based upon recent progress in spectrum analysis and astrophysical theory have been of equal assistance in determining the presence or absence of those elements that have yielded to spectrum analysis.

2. *Absorption Lines and Excitation Potentials.*—Every absorption line is associated with the transition of an atom from one energy state to another of higher energy content, the frequency of the absorbed radiation being exactly proportional to the difference in energy between the two states. An absorption line can be produced only when the atom is already excited to the lower energy state of the transition to which the line is due. The energy of this state is the work required to raise the energy of the atom from its lowest known energy state to the level in which it can absorb the energy $h\nu$ corresponding to the line. Expressed in "volts" it is called the excitation potential, the "volt" being the kinetic energy acquired by an electron in falling through a difference of potential of one volt. For any element the absorption lines arising through transitions from the lowest to the next higher level with which it combines are the ultimate or resonance lines and, provided they are in the region of the spectrum accessible to observation, are the lines most probable of occurrence in the sun. If lines from the lowest level are not within the accessible region, then those from the next higher level are the most probable. The

excitation potential therefore furnishes a criterion upon which to base the probability of occurrence.

3. *Multiplet Relations.*—For many lines the probability of their presence in the sun's spectrum may be estimated from their multiplets. The identification of the stronger lines of the multiplet facilitates the recognition of the weaker members through the prediction of their positions and estimated intensities. A considerable number of Cr^+ and Fe lines predicted from a study of their multiplets, for which the solar wavelengths were used as a source, have been identified with solar lines agreeing with the predicted positions, although the lines in question have not been observed in the laboratory or, if observed, have been too poorly measured.[2] Such lines are indicated in the *Revision* by an appropriately defined symbol.

4. *Behavior in Spots, Stars and the Flash Spectrum.*—For many elements the appearance and relative strength of their lines in the spectra of the disk and spots are characteristic and often furnished definitive information.[3] Lines of elements in the first stage of ionization are weakened in the spectra of spots, but strengthened, relatively to the lines from neutral atoms, in the spectra of higher temperature stars such as γ Cygni.[4] The enhanced lines have great prominence in eclipse spectra of the higher layers of the sun's atmosphere and show identifiable characteristics in flash spectra of the reversing layer.[5]

5. *Pressure and Temperature Classes.*—The negative residuals shown by lines of the pressure group *d* and the abnormally large positive residuals by lines of group *e* for λ sun $-$ λ arc in air, together with the characteristic behavior of lines in furnace spectra as indicated by their temperature classification, have in many cases supplied the deciding criteria. Even with all the present means of identification some identifications must in the last analysis rest upon the acquired experience and the not infallible judgment of the investigator.

B. RESULTS.—Of the 92 possible elements 90 have been found in the earth, and of these the 58 shown in table 1 without parentheses have been identified in the sun's atmosphere. Among the recent identifications interest attaches to atomic carbon, nitrogen, oxygen and sulphur, and to the limitation of lithium, rubidium, indium and boron oxide to the low-temperature region of sun-spots. It is not to be concluded that the unidentified elements are absent from the sun. Elements such as fluorine, chlorine, bromine, iodine, phosphorus and the inert gases, neon, argon, krypton and xenon, have their resonance lines in the ultra-violet beyond the accessible region in solar and stellar spectra and may therefore be present in moderate abundance without the possibility of detection. A layer of ozone some 50 km. above the earth's surface, equivalent to a thickness of 3 mm. at normal pressure, is opaque to a region in the ultra-

violet and limits spectroscopic observation of the radiation from sun and stars to wave-lengths greater than 2975 Å.

It is possible that further investigation may identify one or more of these missing elements for which the lines from the second energy level are in the deep red; and it is quite probable that a number of heavy metals, particularly among the so-called rare-earth elements, may be detected in the sun when more precise laboratory data are available and their spectra have been analyzed. These will be represented by the fainter

TABLE 1

ELEMENTS IDENTIFIED IN THE SUN AND THEIR POSITION IN THE PERIODIC TABLE

( ) UNIDENTIFIED

I	II	III	IV	V	VI	VII	VIII			
H							He			
Li	Be	B	C	N	O	(F)	(Ne)			
Na	Mg	Al	Si	(P)	S	(Cl)	(A)			
K	Ca	Sc	Ti	V	Cr	Mn		Fe	Co	Ni
Cu	Zn	Ga	Ge	(As)	(Se)	(Br)	(Kr)			
Rb	Sr	Y	Zr	Cb?	Mo	(Ma)		Ru	Rh	Pd
Ag	Cd	In	Sn?	Sb	(Te)	(I)	(Xe)			
Cs	Ba	La*	Hf	(Ta)	W	(Re)		(Os)	(Ir)	Pt
(Au)	(Hg)	Tl?	Pb	(Bi)	(Po)	85	(Rn)			
87	(Ra)	(Ac)	(Th)	(Pa)	(U)					

* RARE EARTHS

Ce	Pr	Nd	(Il)	Sa	Eu	Gd
(Tb)	Dy	(Ho)	Er?	(Tu)	Yb?	(Lu)

lines in the sun's spectrum. Lines of intensity 1 to −3 form 82.5 per cent of the total number and furnish 98 per cent of the unidentified lines.

TABLE 2

PERCENTAGE OF IDENTIFICATION FOR INTENSITY GROUPS

INTENSITY	NO. OF LINES	IDENTIFIED PER CENT
8 to 200	227	100
7	65	99
6	170	100
5	309	99
4	534	99
3	959	96
2	1572	92
1 to −3	17917	49

Of note among the strong unidentified lines are 1 of intensity 7, 3 of intensity 5, and 6 of intensity 4.[6] If, as seems quite certain, these strong lines are due to common elements, their identification promises to reveal exceptional conditions of excitation in the sun, as corresponding lines are not produced in the laboratory by present methods of excitation.

C. RÔLE OF EXCITATION POTENTIAL.—The following considerations show the importance of low excitation potentials in the production of solar absorption lines. For 37 metallic elements showing thousands of lines with maximum intensity 40, the average excitation of the lines due to neutral atoms is 1.139 volts, the highest values being 5.771 for lead and 5.467 for manganese. In marked contrast are the average values for non-metals. (Table 3.)

TABLE 3

AVERAGE EXCITATION POTENTIAL FOR NON-METALS

ELEMENT	AVERAGE EXCITATION POTENTIAL	MAX. LINE-INT.	NO. OF LINES
Helium	20.871	0	1
Hydrogen	11.095	40	6
Nitrogen	10.000	—2	1
Oxygen	9.292	2	5
Carbon	7.499	0	9
Sulphur	6.496	—1	3

With the exception of hydrogen, the elements having high excitation potentials are represented in the sun by fewer than a score of weak lines. The extraordinary strength of the hydrogen lines points to the extensive atmosphere of hydrogen shown by eclipse spectra.

The ionized metals illustrate the *inverse* relation between energy level and the percentage occurrence of ionized lines. The energy level for absorption of enhanced lines is the ionization potential plus the excitation potential.

TABLE 4

INCREASE OF PERCENTAGE OF IONIZED LINES WITH DECREASE OF ENERGY LEVEL

QUANTUM ORBIT	ELEMENTS	MEAN ENERGY LEVEL VOLTS	MEAN PER CENT OF IONIZED LINES
3	Mg Al Si	15.10	12.3
4	Ca Sc Ti V Cr	8.59	24.6
5	Sr Y Zr	7.09	47.6
6	Ba La	6.43	100.0

The percentage of lines due to ionized atoms increases progressively with decrease of effective energy level—I. P. *plus* mean E. P.—from the 3-quantum orbit to the 6-quantum orbit. Manganese, iron, cobalt and nickel in the 4-quantum orbit form a restricted group with only three per cent of the lines ionized. The mean energy level of their ionized atoms, 10.08 volts, is higher than that for the other elements in the same quantum orbit, viz., 8.59 volts.

The rare-earth elements appear in the ionized state and presumably have ionization potentials comparable with those of lanthanum and barium. They form a group of particular interest.[7]

The investigation of the chemical constitution of the sun, though greatly advanced, cannot be regarded as completed while there is a fair probability

of the identification of other elements and while a host of faint solar lines, and even a few strong ones, offer their challenge, whose identification may furnish the means of studying exceptional conditions in the sun with fuller understanding of both sun and stars. An outstanding anomaly is the appearance of the ionized helium line λ4686 in the chromosphere. The ionization potential of 24.41 volts and an excitation potential of 48.16 give 72.57 volts required for the production of the line. This exceptionally high value points to some extraordinary condition of excitation in a relatively high layer in the sun's atmosphere and awaits an "explanation."

[1] *Carnegie Institution of Washington, Publication*, No. 396; *Papers of the Mount Wilson Observatory*, **3**, 1928.

[2] Charlotte E. Moore and H. N. Russell, *Mt. Wilson Contr.*, No. 365; *Astrophys. J.*, Chicago, Ill., **68**, 1928 (151–164).

[3] W. S. Adams, *Mt. Wilson Contr.*, No. 40; *Astrophys. J.*, Chicago, Ill., **30**, 1909 (86–126).

[4] W. S. Adams and A. H. Joy, *Mt. Wilson Comm.*, No. 99; these PROCEEDINGS. **13**, 1927 (393–396).

[5] S. A. Mitchell, *Astrophys. J.*, Chicago, Ill., **38**, 1913 (407–495); D. H. Menzel, *Publ. Astr. Soc. Pacific*, San Francisco, Calif., **39**, 1927 (359–360).

[6] *Revision*, p. 205.

[7] C. E. St. John and Charlotte E. Moore, *Mt. Wilson Contr.*, No. 364; *Astrophys J.*, Chicago, Ill., **68**, 1928 (93–108).

BIOMETRIC FUNCTIONS IN A POPULATION GROWING IN ACCORDANCE WITH A PRESCRIBED LAW

BY ALFRED J. LOTKA

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1. *Births.*—The individuals in a population, who at time t are a years old, are the survivors of those born at time $t - a$; and, in the absence of immigration and emigration, the total number of the population, at time t, is the sum or integral formed for persons thus surviving to all ages. If we denote by B_t the number of births per unit of time at time t, by $p(a)$ the probability, at birth, of surviving to age a, and by N_t the population at time t, we have the fundamental relation

$$\int_0^\infty B_{t-a}p(a)da = N_t. \qquad (1)$$

If B_t and $p(a)$ are known functions, N_t is immediately determined by the integral on the left-hand side of this equation. But the converse problem of determining B_t when $p(a)$ and N_t are given functions requires special treatment.

Case 1.—The simplest case is that of a stationary population. Here we have

$$\int_0^\infty B_{t-a}p(a)da = \text{const.} \tag{2}$$

a solution of which (when $p(a)$ is prescribed) is seen by inspection, namely,

$$B_t = N/\int_0^\infty p(a)da = N/L_0 = \text{const.} \tag{3}$$

where L_0 is the mean length of life.

Case 2.—Another very simple case is that of a population growing according to the law of Malthus, i.e.,

$$\int^\infty B_{t-a}p(a)da = N_0\epsilon^{rt}. \tag{4}$$

Again, by inspection, we find a solution

$$B_t = B_0\epsilon^{rt} = N_0\epsilon^{rt}/\int_0^\infty \epsilon^{-ra}p(a)da. \tag{5}$$

Case 3.—The last case suggests at once a generalization of wide scope. For if N_t is any arbitrary function that can be represented as a series $\sum U_k\epsilon^{r_k t}$ (a supposition that allows great latitude), we have

$$\int_0^\infty B_{t-a}p(a)da = U_1\epsilon^{r_1 t} + U_2\epsilon^{r_2 t} + \ldots, \tag{6}$$

of which, again by inspection, we find a solution

$$B_t = V_1\epsilon^{r_1 t} + V_2\epsilon^{r_2 t} + \ldots, \tag{7}$$

where

$$V_k = \frac{U_k}{\int_0^\infty -\epsilon^{-r_k a}p(a)da}. \tag{8}$$

Case 4.—A special case arises if $r_k = kr$. Then

$$B_t = V_1\epsilon^{rt} + V_2\epsilon^{2rt} + \ldots \tag{9}$$

$$V_k = \frac{U_k}{\int_0^\infty \epsilon^{-k}\ p(a)da}. \tag{10}$$

Case 5.—A particular interest attaches to the case of a population growing according to the Verhulst-Pearl law. Here we have

$$\int_0^\infty B_{t-a}p(a)da = \frac{N_\infty}{1+\epsilon^{-r_i t}} = \frac{N_\infty\epsilon^{r_i t}}{1+\epsilon^{r_i t}}, \tag{11}$$

where r_i is the *incipient* rate of increase.

By a formal transformation the right-hand member is converted into a geometric series, so that

$$\int_0^\infty B_{t-a}p(a)da = N_\infty(\epsilon^{r_i t} - \epsilon^{2r_i t} + \epsilon^{3r_i t} - \epsilon^{4r_i t} + \ldots), \tag{12}$$

and the solution is

$$B_t = V_1\epsilon^{r_i t} + V_2\epsilon^{2r_i t} + \ldots, \tag{13}$$

where

$$V_k = \frac{(-1)^{k-1}N_\infty}{\int^\infty \epsilon^{-kr_i a}p(a)da}. \tag{14}$$

We will write, for brevity,

$$\int^\infty \epsilon^{-kr_i a}p(a)da = \frac{1}{b_k}. \tag{15}$$

Then

$$B_t = N_\infty b_1\epsilon^{r_i t}\left(1 - \frac{b_2}{b_1}\epsilon^{r_i t} + \frac{b_3}{b_1}\epsilon^{2r_i t} - \ldots\right) \tag{16}$$

$$= \frac{N_\infty b_1\epsilon^{r_i t}}{1 + P_1\epsilon^{r_i t} + P_2\epsilon^{2r_i t} + \ldots}, \tag{17}$$

where

$$b_1P_k = b_2P_{k-1} - b_3P_{k-2} + b_4P_{k-3} - \ldots. \tag{18}$$

For the birthrate *per head* per unit of time we find, according to (17) and (11),

$$b_i = \frac{B_t}{N_t} = \frac{b_1(1 + \epsilon^{r_i t})}{1 + P_1\epsilon^{r_i t} + P_2\epsilon^{2r_i t} + \ldots}. \tag{19}$$

Or, according to (16) and (11) we have

$$b_i = \frac{B_t}{N_t} = (1 + \epsilon^{r_i t})\,(b_1 - b_2\epsilon^{r_i t} + b_3\epsilon^{2r_i t} - \ldots) \tag{20}$$

$$= (1 + \epsilon^{r_i t})\sum(-1)^{k-1}b_k\epsilon^{(k-1)r_i t}. \tag{21}$$

For purposes of computation it is desirable to cast (20) into another form. We introduce the notation

$$L_k = \int_0^\infty a^k p(a)da. \tag{22}$$

Then, expanding the exponential function under the integral sign in (15) we find

$$\frac{1}{b_k} = L_0 - kr_i L_1 + k^2\frac{r_i^2}{2!}L_2 - \ldots \tag{23}$$

$$b_k = c_0 + c_1kr_i + c_2k^2r_i^2 + c_3k^3r_i^3 + \ldots \tag{24}$$

where

$$\left.\begin{aligned} c_0 &= \frac{1}{L_0};\ c_1 = \frac{L_1}{L_0^2};\ c_2 = \frac{L_1^2}{L_0^3} - \frac{L_2}{2L_0^2};\\ c_3 &= \frac{L_1^3}{L_0^4} - \frac{L_1L_2}{L_0^3} + \frac{L_3}{6L_0^2};\\ c_4 &= \frac{L_1^4}{L_0^5} - \frac{3L_1^2L_2}{2L_0^4} + \frac{L_1L_3}{3L_0^3} + \frac{L_2^2}{4L_0^3} - \frac{L_4}{24L_0^2};\\ c_5 &= \frac{L_1^5}{L_0^6} - \frac{2L_1^3L_2}{L_0^5} + \frac{L_1^2L_3}{2L_0^4} + \frac{3L_1L_2^2}{4L_0^4}\\ &\quad - \frac{L_2L_3}{6L_0^3} - \frac{L_1L_4}{12L_0^3} + \frac{L_5}{120L_0^2}, \end{aligned}\right\} \quad (25)$$

and finally, according to (21), (24), (25)

$$\left.\begin{aligned} b_t &= c_0 + c_1\frac{r_i}{1+\epsilon^{r_it}} + c_2\frac{r_i^2}{(1+\epsilon^{r_it})^2}(1-\epsilon^{r_it})\\ &+ c_3\frac{r_i^3}{(1+\epsilon^{r_it})^3}(1-4\epsilon^{r_it}+\epsilon^{2r_it})\\ &+ c_4\frac{r_i^4}{(1+\epsilon^{r_it})^4}(1-11\epsilon^{r_it}+11\epsilon^{2r_it}-\epsilon^{3r_it})\\ &+ c_5\frac{r_i^5}{(1+\epsilon^{r_it})^5}(1-26\epsilon^{r_it}+66\epsilon^{2r_it}-26\epsilon^{3r_it}+\epsilon^{4r_it})\\ &+ \cdots \end{aligned}\right\} \quad (26)$$

$$= \frac{1}{\int_0^\infty \epsilon \exp\left(-\frac{r_ia}{1+\epsilon^{r_it}}\right)p(a)da} + R, \quad (27)$$

where R, is, in practice, relatively small, so that a fair approximation is obtained from the first term of (27) alone.

It is to be noted that $\frac{r_i}{1+\epsilon^{r_it}}$ is the momentary value of the Verhulst-Pearl rate of increase at time t. If we indicate this by the notation

$$r_t = \frac{r_i}{1+\epsilon^{r_it}}. \quad (28)$$

(27) may also be written

$$b_t = \frac{1}{\int_0^\infty \epsilon^{-r_ta}p(a)da} + R. \quad (29)$$

An alternative expansion, which may be preferred for computation when $t > 0$, is obtained by dividing numerator and denominator of each term in (26) by the highest power of $e^{r_i t}$ occurring in that term. We thus obtain

$$\left.\begin{aligned} b_t = c_0 + c_1 r_i \frac{\epsilon^{-r_i t}}{(1+\epsilon^{-r_i t})} - c_2 r_i^2 \epsilon^{-r_i t} \frac{(1-\epsilon^{-r_i t})}{1+\epsilon^{-r_i t})^2} \cdot \\ + c_3 r_i^3 \epsilon^{-r_i t} \frac{(1-4\epsilon^{-r_i t}+\epsilon^{-2r_i t})}{(1+\epsilon^{-r_i t})^3} +, \text{ etc.} \end{aligned}\right\} \quad (30)$$

Extreme Values of b_t.—When $t = -\infty$ the remainder term in (29) evidently vanishes, and we have therefore

$$b_i = \frac{1}{\int_0^\infty \epsilon^{-r_i a} p(a) da} \quad (31)$$

$$= c_0 + c_1 r_i + c_2 r_i^2 + c_3 r_i^3 + \ldots, \quad (32)$$

where r_i denotes the *incipient* rate of increase, and b_i the corresponding *incipient* birthrate, that is, the value of the birthrate per head when the population consists of a very small nucleus.

When $t = +\infty$ we readily find, by (30)

$$b_\infty = c_0 = \frac{1}{L_0}, \quad (33)$$

that is, the ultimate birthrate is the reciprocal of the mean length of life according to the life table at the basis of the argument. This, of course, is as it should be.

2. *Age Distribution.*—When B_t, the total number of births per unit of time at time t, has been determined as above, the age distribution follows at once; for, the $N_t\, c(a)da$ persons comprised, at time t, within the age limits a and $a + da$, are the survivors $B_{t-a}p(a)da$ of the persons born at time $(t - a)$ in an interval da, that is

$$N_t c(a) da = B_{t-a} p(a) da \quad (34)$$

$$c(a) = \frac{B_{t-a}}{N_t}\,(pa). \quad (35)$$

3. *Fertility.*—Since the life table function $p(a)$ has here been considered constant, the diminishing rate of increase characteristic of the Verhulst-Pearl law must in this case result from a diminishing fertility. We may consider the nature of this decline as follows:

It will be convenient here to center our attention on the female popu-

lation. Let $m(a)$ be the maternity frequency of mothers of age a, counting births of daughters only. Then, evidently

$$B_t = \int_0^\infty B_{t-a}p(a)m_t(a)da \tag{36}$$

$$= \bar{m}_t \int_u^v B_{t-a}p(a)da. \tag{37}$$

Here $\bar{m}_t$ is a species of average measure of net fertility, which may serve to compare fertility at successive epochs t. The limits of u, v of the integral are the lower and upper limit of the reproductive period, say ages 15 and 55.

If we make the assumption that the frequency function $m(a)$ preserves its form, and changes only its amplitude, so that

$$m_t(a) = \lambda_t m(a) \tag{38}$$

we can also write (36)

$$B_t = \lambda_t \int_0^\infty B_{t-a}p(a)m(a)da \tag{39}$$

and λ_t may then be used as a measure of the comparative fertility at successive epochs.

4. *Inherent Rate of Natural Increase.*—Corresponding to any value λ_t defining the fertility at time t, a value r_λ for the *inherent* rate of natural increase (i.e., the rate which is permanently compatible with that particular fertility and with $p(a)$, the prevailing survival function) is obtained from the fundamental relation

$$1 = \lambda_t \int_0^\infty \epsilon^{-r\lambda a}p(a)m(a)da. \tag{40}$$

5. *Time-Trend of the Life Table Function $p(a)$.*—A rigorous treatment of the problem arising from the time-trend (secular improvement) of the life table (survival) function $p(a)$ will not here be offered. The relation (29) suggests that a representation for the case of a *fluent* life table may be obtained in the form

$$b_t = \frac{1}{\int_0^\infty \epsilon^{-r_t a}p_{t-a}(a)da} + R', \tag{41}$$

and that as a first approximation the term R' may be omitted. We shall not here follow this suggestion into further detail.

BOUNDARY VALUES OF AN ANALYTIC FUNCTION AND THE TCHEBYCHEFF METHOD OF APPROXIMATION

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If the functions $f(z)$, $f_n(z)$, $n = 1, 2, \ldots$, are analytic on and within the unit circle C of the z-plane, the relation

$$\lim_{n \to \infty} [\max | f(z) - f_n(z) |] = 0, \; z \text{ on } C, \tag{1}$$

implies the relation

$$\lim_{n \to \infty} f_n(z) = f(z) \tag{2}$$

uniformly for z on and within C.

There are, however, cases of approximation formally similar to this—which consist, namely, of approximation by analytic functions to a function defined and continuous on the boundary C of a region B—where approximation with an arbitrarily small error is not possible. For instance, the function $f(z)$ to be approximated may be the boundary values of an analytic function *with singularities in* B, or the approximating functions $f_n(z)$ may all be required to take on prescribed values at definite points interior to B; in neither of these cases is (1) possible in general. Natural questions to be raised in these new cases are: 1) What is the minimum value m (or greatest lower bound) of

$$\max | f(z) - f_n(z) | , \; z \text{ on } C,$$

for all functions $f_n(z)$ analytic interior to C and taking on the prescribed values if any? 2) Does a function $f_o(z)$, taking on the prescribed values if any, exist such that

$$\max | f(z) - f_o(z) | = m, \text{ for } z \text{ on } C \text{ ?}$$

3) If we have

$$\lim_{n \to \infty} [\max | f(z) - f_n(z) | , \; z \text{ on } C] = m,$$

where the functions $f_n(z)$ are analytic interior to C and take on the prescribed values if any, what can be said of the convergence of the sequence $\{f_n(z)\}$? 4) In particular, if we consider as a special sequence the Tchebycheff polynomials for approximation to $f(z)$ on C, what can be said of the convergence?

There are two theorems of primary interest which answer these questions:

THEOREM I. *Let B be an arbitrary simply connected region with the boundary C, and let the function $f(z)$ be analytic on C, meromorphic on $B + C$. Then there exists a function $f_o(z)$ analytic interior to B such that*[1]

$$\overline{\text{bound}}_{z \to C} \left| f(z) - f_o(z) \right| = m \tag{3}$$

is less than

$$\overline{\text{bound}}_{z \to C} \left| f(z) - f_n(z) \right| \tag{4}$$

for any other function $f_n(z)$ analytic interior to B. The expression $|f(z) - f_o(z)|$ approaches the same limit m no matter how z in B approaches C. If the functions $f_n(z)$, $n = 1, 2, \ldots$, are analytic interior to B, then

$$\lim_{n \to \infty} \left[\overline{\text{bound}}_{z \to C} \left| f(z) - f_n(z) \right| \right] = m$$

implies

$$\lim_{n \to \infty} f_n(z) = f_o(z)$$

for z in B, uniformly for z on an arbitrary closed point set interior to B.

If the region B is limited and if its boundary C is also the boundary of an infinite region, the Tchebycheff polynomial $\omega_n(z)$ of degree n for approximation to $f(z)$ on C is defined as the polynomial of degree n such that

$$\max \left| f(z) - \omega_n(z) \right|, \; z \text{ on } C,$$

is less than the corresponding expression formed for any other polynomial of degree n; the Tchebycheff polynomial $\omega_n(z)$ exists and is unique. It is then true if $f(z)$ has no singularities other than in B and in the infinite region of which C is the boundary, that

$$\lim_{n \to \infty} \left[\max \left| f(z) - \omega_n(z) \right|, \; z \text{ on } C \right] = m,$$

from which it follows that

$$\lim_{n \to \infty} \omega_n(z) = f_o(z) \tag{5}$$

for z in B, uniformly on any closed point set interior to B.

If B is the interior of a limited analytic curve C, we denote by C_R the curve which is the locus $|\varphi(z)| = R$, where $w = \varphi(z)$ is a function which maps the exterior of C onto the exterior of the unit circle $|w| = 1$ so that the points at infinity correspond to each other. In this case we have

$$\lim_{n \to \infty} \omega_n(z) = f_o(z) \tag{5}$$

not merely for z in B but also for z interior to the largest curve C_R which contains in its interior no singularity of $f_o(z)$—such a curve C_R exists and

is exterior to C—and the convergence in (5) is uniform on an arbitrary closed point set interior to this C_R.

In the second theorem we shall be concerned with approximation on the boundary C of a region B by functions which satisfy certain auxiliary conditions at a finite number of preassigned points P interior to B. These conditions are, at each point P, prescription of the value of the function together with the values of its first k_P derivatives.

THEOREM II. *Let B be an arbitrary simply connected region with the boundary C, and let the function* $F(z)$ *be analytic on* $B + C$. *Then there exists a function* $F_0(z)$ *analytic interior to B satisfying arbitrary given auxiliary conditions, such that*

$$m = \overline{\text{bound}} \mid F(z) - F_0(z) \mid, z \text{ in } B,$$

is less than

$$\overline{\text{bound}} \mid F(z) - F_n(z) \mid, z \text{ in } B,$$

for any other function $F_n(z)$ *analytic interior to B and satisfying the same auxiliary conditions. The expression* $\mid F(z) - F_0(z) \mid$ *approaches the same limit m no matter how z in B approaches C. If the functions* $F_n(z)$, $n = 1, 2, \ldots$, *are analytic interior to B and satisfy the given auxiliary conditions, then*

$$\lim_{n \to \infty} [\overline{\text{bound}} \mid F(z) - F_n(z) \mid, z \text{ in } B] = m$$

implies

$$\lim_{n \to \infty} F_n(z) = F_0(z)$$

for z in B, uniformly for z on arbitrary closed point set interior to B.

If the region B is limited and if its boundary C is also the boundary of an infinite region, the Tchebycheff polynomial $\pi_n(z)$ *of degree n for approximation to* $F(z)$ *on C with the given auxiliary conditions is the polynomial of degree n which satisfies the auxiliary conditions such that*

$$\max \mid F(z) - \pi_n(z) \mid, z \text{ on } C,$$

is less than the corresponding expression formed for any other polynomial of degree n and satisfying the auxiliary conditions; this Tchebycheff polynomial $\pi_n(z)$ *exists for n sufficiently large and is unique. It is true if* $F(z)$ *has no singularities other than in the infinite region bounded by C that*

$$\lim_{n \to \infty} [\max \mid F(z) - \pi_n(z) \mid, z \text{ on } C] = m,$$

from which it follows that

$$\lim_{n \to \infty} \pi_n(z) = F_0(z) \tag{6}$$

for z in B, uniformly on any closed point set interior to B.

If B is the interior of a limited analytic curve C and if C_R has the same generic meaning as before, we have (6) *valid not merely for z in B but also for z interior to the largest curve C_R which contains in its interior no singularity of $F_0(z)$—such a curve exists and is exterior to C—and the convergence in* (6) *is uniform on an arbitrary closed point set interior to this C_R.*

These two theorems and generalizations can be proved by methods and results due to Carathéodory and Fejér, Gronwall, Schur, R. Nevanlinna, Carathéodory, and Tonelli. Some of the results of Theorem II have recently been obtained by Julia, for special auxiliary conditions of the form $F(O) = 0$, $F'(O) = 1$. This case is connected with the conformal mapping of the region B onto a circle, as is the case of Theorem I for which $f(z)$ is of the form $1/z$. Detailed references to the literature and detailed proofs of Theorems I and II will appear later in another journal.

[1] The notation here is meant to imply that we allow z in B to approach C in any way whatever, and take the least upper bound of all corresponding limits of $|f(z) - f_0(z)|$.

ON THE SPHERICALLY SYMMETRICAL STATICAL FIELD IN EINSTEIN'S UNIFIED THEORY: A CORRECTION

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In a previous paper[1] the authors of the present note have treated the case of a spherically symmetrical statical field, and stated the conclusions: first, that under Einstein's definition of the electromagnetic potential an electromagnetic *field* is incompatible with the assumptions of static spherical symmetry and of symmetry of past and future; second, that if one uses the Hamiltonian suggested in Einstein's second 1928 paper[2] the electromagnetic potential vanishes and the gravitational field also vanishes. We have recently become aware that the method used in that paper is incorrect because the definition of $\Lambda^{\nu}_{\mu\sigma}$ given on page 353 is valid only if the coördinates in the local axes are taken as Cartesian. In this note we replace the reasoning of the previous paper by an argument employing Cartesian local coördinate axes and Cartesian Gaussian coördinates in the continuum. *None of the conclusions of the previous paper are vitiated by this investigation*, although some of the final formulas are supplemented by an additional term.

Let d^1x, d^2x, d^3x, d^4x be the elements of length along the axes of the local quadruple ("4-Bein"), x, y, z, t, the Gaussian coördinates. We

retain the assumptions and notations of the previous paper but introduce the abbreviations $(U-1)/\Lambda^2 = A$, $M/\Lambda = B$, $N/\Lambda = C$. Then

$$\begin{aligned} d^1x &= dx + Ax(x\,dx + y\,dy + z\,dz) + Bxdt \\ d^2x &= dy + Ay(x\,dx + y\,dy + z\,dz) + Bydt \\ d^3x &= dz + Az(x\,dx + y\,dy + z\,dz) + Bzdt \\ d^4x &= Cxdx + Cydy + Czdz + Wdt. \end{aligned}$$

The covariant components ${}^ah_\lambda$ therefore become

$$\begin{array}{llll} {}^1h_1 = 1 + Ax^2 & {}^1h_2 = Axy & {}^1h_3 = Axz & {}^1h_4 = Bx \\ {}^2h_1 = Axy & {}^2h_2 = 1 + Ay^2 & {}^2h_3 = Ayz & {}^2h_4 = By \\ {}^3h_1 = Axz & {}^3h_2 = Ayz & {}^3h_3 = 1 + Az^2 & {}^3h_4 = Bz \\ {}^4h_1 = Cx & {}^4h_2 = Cy & {}^4h_3 = Cz & {}^4h_4 = W \end{array}$$

The determinant $D = |{}^ah_\lambda| = UW - MN$ as before. The contravariant components ${}_ah^\lambda$ are [putting $E = (BC - AW)/D$]

$$\begin{array}{llll} {}_1h^1 = 1 + Ex^2 & {}_1h^2 = Exy & {}_1h^3 = Exz & {}_1h^4 = -Cx/D \\ {}_2h^1 = Exy & {}_2h^2 = 1 + Ey^2 & {}_2h^3 = Eyz & {}_2h^4 = -Cy/D \\ {}_3h^1 = Exz & {}_3h^2 = Eyz & {}_3h^3 = 1 + Ez^2 & {}_3h^4 = -Cz/D \\ {}_4h^1 = -Bx/D & {}_4h^2 = -By/D & {}_4h^3 = -Bz/D & {}_4h^4 = (1 + Ar^2)/D \end{array}$$

The components $\varphi_\mu = \Lambda^\alpha_{\mu\alpha}$ of the electromagnetic potential are

$$\begin{aligned} \varphi_1 &= \frac{x}{2}\left(\frac{2(U-1)}{r^2} + \frac{M'N - UW'}{Dr}\right) \\ \varphi_2 &= \frac{y}{2}\left(\frac{2(U-1)}{r^2} + \frac{M'N - UW'}{Dr}\right) \\ \varphi_3 &= \frac{z}{2}\left(\frac{2(U-U}{r^2} + \frac{M'N - UW'}{Dr}\right) \\ \varphi_4 &= \frac{M}{r} + \frac{1}{2}\frac{M'W - MW'}{D} \end{aligned}$$

Returning to polar coördinates, we obtain

$$\begin{aligned} \varphi_r &= \frac{U-1}{r} + \frac{1}{2}\frac{M'N - UW'}{D} \quad (\varphi_1 \text{ of previous paper}) \\ \varphi_\theta &= \varphi_\varphi = 0 \\ \varphi_t &= \frac{M}{r} + \frac{1}{2}\frac{WM' - MW'}{D} \quad (\varphi_4 \text{ of previous paper}). \end{aligned}$$

These differ from the expressions of the last paper essentially only in the

first terms, although there is also an error of sign in the second term of the previous paper due to a mistake in the signs of ${}_4h^1$ and ${}_1h^4$ on p. 354. Under the assumption of symmetrical time $M = N = 0$, and our potential reduces to

$$\varphi_r = \frac{U-1}{r} - \frac{1}{2}\frac{\partial}{\partial r}\log W$$
$$\varphi_\vartheta = \varphi_\varphi = \varphi_t = 0.$$

Thus the electromagnetic *field* vanishes.

We shall now show that if we use the Hamiltonian $H = g^{\lambda\mu}\Lambda^{\alpha}_{\lambda\beta}\Lambda^{\beta}_{\mu\alpha}$ of Einstein's second 1928 paper, it follows from the field equations that $U = 1$ and $W = \text{const.}$ so that the electromagnetic potential vanishes. On the assumptions of symmetry of past and future the Hamiltonian becomes

$$H = \frac{1}{4U^2}\left[\frac{2(U-1)^2}{r^2} + \frac{W'^2}{W^2}\right]$$

and the field equations are obtained from the variational equations

$$\delta\int^{\infty} r^2 DH dr = 0 \qquad (D = UW).$$

Whether W is real or imaginary, W'/W is real, so that the expression within the brackets is the sum of two squares of real quantities. Furthermore, U and W do not pass through zero in any physically significant case. Thus the condition for an absolute minimum is $U = 1$, $W = \text{const.}$, and all components of the electromagnetic potential vanish.

In the absence of time symmetry the tensor $\Lambda^{\mu}_{\gamma\delta}$ possesses in general 36 non-vanishing Cartesian components. The condition of time symmetry reduces this number to 18. If now $U = 1$, $W = \text{const.}$, all vanish and the space is Euclidean. In this case the gravitational field vanishes as well.

[1] *Proc. Nat. Acad. Sci.*, **15**, 353, 1929.

[2] A. Einstein, *Berliner Berichte*, **1928**, 224–227.